$$= \frac{\theta^{r+\alpha-1} e^{-\theta\beta}[\beta + \sum ln(1+y_i{}^c) + (n-r)\, ln(1+t_0{}^c)]^{r+\alpha}}{(1+t_0{}^c)^{\theta(n-r)}(\prod_{i=1}^{n}(1+y_i{}^c))^{\theta}\Gamma(\alpha+r)} \quad (11)$$

Under squared error loss, the Bayes estimator of θ is the posterior mean given by

$$\hat{\theta}_2 = E\left(\theta \middle| \underline{y}\right) = \int_0^\infty \theta\, \pi(\theta|\underline{y})\, d\theta$$

$$= \frac{\alpha + r}{\beta + \sum ln(1+y_i{}^c) + (n-r)\, ln(1+t_0{}^c)} \quad (12)$$

when $\alpha = 0$ *and* $\beta = 0$, *then* equation (12) becomes

$$\hat{\theta}_2 = \frac{r}{\sum ln(1+y_i{}^c) + (n-r)\, ln(1+t_0{}^c)}$$

i.e., the same as MLE.

The Bayes estimator of reliability is

$$\hat{R}_2 = E\left[(1+t^c)^{-\theta} \middle| \underline{y}\right]$$

$$= \int_0^\infty (1+t^c)^{-\theta}\, \pi\left(\theta\middle|\underline{y}\right) d\theta$$

$$= [\frac{\beta + \sum ln(1+y_i{}^c) + (n-r)\, ln(1+t_0{}^c)}{\beta + \sum ln(1+y_i{}^c) + (n-r)\, ln(1+t_0{}^c) + ln(1+t^c)}]^{r+\alpha} \quad (13)$$

3. <u>The case of partial ordering *I*</u>

Assume that we observe the lifetime of the units which failed in the interval $(0, t_0]$ and obtain the number of failures in the interval $(t_0, t_1]$. Let $\underline{y} = (y_1, \ldots, y_r)$, the conditional joint probability density function (jpdf) of the observed sample is given by

$$L\left(\underline{y}\middle|\theta\right)$$

$$= f(r,k) f\left(\underline{y}\middle|R = r, K = k, \theta\right) \quad (14)$$

$$= \frac{n!}{k!\,(n-r-k)!}[F(t_0)]^r[F(t_1) - F(t_0)]^k[1 - F(t_1)]^{n-r-k}\prod_{i=1}^{r}\frac{f(y_i)}{F(t_0)}$$

$$= \frac{n![(1+t_0{}^c)^{-\theta} - (1+t_1{}^c)^{-\theta}]^k[1+t_1{}^c]^{-\theta(n-r-k)}c^r\theta^r(\prod_{i=1}^{r} y_i)^{c-1}}{(n-r-k)!k!\,(\prod_{i=1}^{r}(1+y_i{}^c))^{\theta+1}}$$

Then the pdf of the posterior is

$$\pi\left(\theta\middle|\underline{y}\right) = \frac{L(\underline{y}|\theta)\pi(\theta)}{\int_0^\infty L\left(\underline{y}\middle|\theta\right)\pi(\theta)d\theta} \quad (15)$$

$$= \frac{[(1+t_0{}^c)^{-\theta} - (1+t_1{}^c)^{-\theta}]^k\theta^{r+\alpha-1}e^{-\theta\beta}}{[1+t_1{}^c]^{\theta(n-r-k)}[\prod 1 + y_i{}^c]^{\theta}\int_0^\infty \frac{[(1+t_0{}^c)^{-\theta} - (1+t_1{}^c)^{-\theta}]^k\theta^{r+\alpha-1}e^{-\theta\beta}}{([1+t_1{}^c]^{\theta(n-r-k)}[\prod 1+y_i{}^c]^{\theta})^{\theta}}\, d\theta}$$

Now

$$\int_0^\infty \frac{[(1+t_0{}^c)^{-\theta} - (1+t_1{}^c)^{-\theta}]^k\theta^{r+\alpha-1}e^{-\theta\beta}}{([1+t_1{}^c]^{\theta(n-r-k)}[\prod 1 + y_i{}^c]^{\theta})^{\theta}}\, d\theta$$

$$= \int_0^\infty [(1+t_0{}^c)^{-\theta} - (1+t_1{}^c)^{-\theta}]^k\theta^{r+\alpha-1}\, e^{-\theta[\beta+(n-r-k)\ln(1+t_1{}^c)+\sum ln(1+y_i{}^c)]}\, d\theta$$

$$= \int_0^\infty e^{r+\alpha-1}\sum_{j=0}^{k}(-1)^j(1+t_1{}^c)^{-j\theta}(1+t_0{}^c)^{-(k-j)\theta}e^{-\theta[\beta+(n-r-k)\,ln(1+t_1{}^c)+\sum ln(1+y_i{}^c)]}d\theta$$

$$= \sum_{j=0}^{k}(-1)^j\int_0^\infty \theta^{r+\alpha-1}e^{-\theta A_j}\, d\theta$$

$$= \sum_{j=0}^{k}(-1)^j\frac{\Gamma(r+\alpha)}{A_j{}^{r+\alpha}} \quad (16)$$

where

$$A_j = \beta + (n-r-k)\, ln(1+t_1{}^c) + \sum ln(1+y_i{}^c) + (k-j)\ln(1+t_0{}^c)$$

Therefore, equation (15) becomes

$$\pi\left(\theta\middle|\underline{y}\right)$$

$$= \frac{[(1+t_0{}^c)^{-\theta} - (1+t_1{}^c)^{-\theta}]^k\theta^{r+\alpha-1}e^{-\theta\beta}}{[1+t_1{}^c]^{\theta(n-r-k)}[\prod(1+y_i{}^c)]^{\theta}\sum_{j=0}^{k}(-1)^j\frac{\Gamma(r+\alpha)}{A_j{}^{r+\alpha}}}$$

Under squared error loss, the Bayes estimate of the parameter θ is the posterior mean given by

$$\hat{\theta}_3 = E\left(\theta\middle|\underline{y}\right) = \int_0^\infty \theta\,\pi\left(\theta\middle|\underline{y}\right)d\theta$$

$$= (r+\alpha)\frac{\sum_{j=0}^{k}\frac{(-1)^j}{A_j{}^{r+\alpha+1}}}{\sum_{j=0}^{k}\frac{(-1)^j}{A_j{}^{r+\alpha}}} \quad (18)$$

and the Bayes estimate of R is

$$\hat{R}_3 = \int_0^\infty [1 + t^c]^{-\theta}\pi\left(\theta\middle|\underline{y}\right)\, d\theta \quad (19)$$

$$= \int_0^\infty \frac{[(1+t_0{}^c)^{-\theta} - (1+t_1{}^c)^{-\theta}]^k\theta^{r+\alpha-1}e^{-\theta\beta}}{[1+t^c]^{\theta}[1+t_1{}^c]^{\theta(n-r-k)}[\prod(1+y_i{}^c)]^{\theta}\sum_{j=0}^{k}(-1)^j\frac{\Gamma(r \ldots}{A_j \ldots}}$$

$$= \frac{1}{\Gamma(r+\alpha)\sum_{j=0}^{k}\frac{(-1)^j}{A_j{}^{r+\alpha}}}\int_0^\infty \frac{\sum_{j=0}^{k}(-1)^j(1+t_1{}^c)^{-j\theta}(1+t_0{}^c)^{-(k \ldots}}{[1+t^c]^{\theta}[1+t_1{}^c]^{\theta(n-r-k)}[\prod \ldots}$$

$$= \frac{1}{\Gamma(r+\alpha)\sum_{j=0}^{k}\frac{(-1)^j}{A_j{}^{r+\alpha}}}\int_0^\infty \sum_{j=0}^{k}(-1)^j\theta^{r+\alpha-1}e^{-\theta B_j}\, d\theta$$

$$= \frac{\sum_{j=0}^{k}\frac{(-1)^j}{B_j{}^{r+\alpha}}}{\sum_{j=0}^{k}\frac{(-1)^j}{A_j{}^{r+\alpha}}} \quad (20)$$

where

$$A_j = \beta + (n-r-k)\, ln(1+t_1{}^c) + \sum ln(1+y_i{}^c) + (k-j)\ln(1+t_0{}^c)$$

$$B_j = \beta + (n + j - r - k)\, ln(1 + {t_1}^c) + (k - j)\, ln(1 + {t_0}^c) + \sum ln(1 + {y_i}^c) + \ln(1 + t^c)$$

4. <u>Case of type II censoring</u>

Observe the first r failures $Y_1 \le Y_2 \le \cdots \le Y_r$. The posterior distribution is

$$\pi\left(\theta \middle| \underline{y}\right) = \frac{f(\underline{y}|\theta)\pi(\theta)}{\int_0^\infty f\left(\underline{y}\middle|\theta\right)\pi(\theta) d\theta} \qquad (21)$$

where

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \theta^{\alpha-1}\, e^{-\theta\beta}, \quad \theta, \alpha, \beta > 0$$

$$f\left(\underline{y}\middle|\theta\right) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(y_i)\ [1 - F(y_r)]^{n-r} \qquad (22)$$

$$= \frac{n!}{(n-r)!}\, \frac{c^r \theta^r (\prod_{i=1}^r y_i)^{c-1} (1 + {y_r}^c)^{-\theta(n-r)}}{(\prod_{i=1}^r (1 + {y_i}^c))^{\theta+1}}, \quad 0 \le Y_1 \le Y_2 \le \cdots \le Y_r$$

therefore,

$$\pi\left(\theta \middle| \underline{y}\right) = \frac{\theta^{r+\alpha-1} (1 - {y_r}^c)^{-\theta(n-r)} e^{-\theta\beta} [\beta + (n-r)\, ln(1 + {y_r}^c) + \sum ln(1 + {y_i}^c)]^{r+\alpha}}{\Gamma(r+\alpha)[\prod(1 + {y_i}^c)]^\theta}$$

Under squared error loss, the Bayes estimator of the parameter θ is the posterior mean and is given by

$$\hat{\theta}_4 = E\left(\theta \middle| \underline{y}\right) = \int_0^\infty \theta\, \pi\left(\theta \middle| \underline{y}\right) d\theta$$

$$= \frac{r + \alpha}{\beta + (n-r)\, ln(1 + {y_r}^c) + \sum ln(1 + {y_i}^c)} \qquad (24)$$

And the Bayes estimator of reliability is

$$\hat{R}_4 = \int_0^\infty [1 + t^c]^{-\theta} \pi\left(\theta \middle| \underline{y}\right)\, d\theta$$

$$= \left[\frac{\beta + (n-r)\, ln(1 + {y_r}^c) + \sum ln(1 + {y_i}^c)}{\beta + (n-r)\, ln(1 + {y_r}^c) + \sum ln(1 + {y_i}^c) + ln(1 + t^c)}\right]^{r+\alpha}$$

5. <u>The case of partial ordering II</u>

Assume that we obtain k_1 be the number of failures observed in the interval $(0, t_0]$ and obtain k_2 be the number of failures observed in the interval $(t_0, t_1] mi.e.,$

k_1 = number of observed in $(0, t_0]$

k_2 = number of observed in $(t_0, t_1]$

$k_3 = n - k_1 - k_2$.

Let

$$p_1 = P(x \le t_0) = F_x(t_0) = 1 - (1 + {t_0}^c)^{-\theta}$$

$$p_2 = P(t_0 < x \le t_1) = F_x(t_1) - F_x(t_0) = (1 + {t_0}^c)^{-\theta} - (1 + {t_1}^c)^{-\theta}$$

$$p_3 = P(x > t_1) = 1 - P(x \le t_1) = 1 - F_x(t_1) = (1 + {t_1}^c)^{-\theta}$$

The likelihood function is given by

$$L(k_1, k_1|\theta) = \frac{n!}{k_1!\, k_2!\, k_3!} [1 - (1 + {t_0}^c)^{-\theta}]^{k_1} [(1 + {t_0}^c)^{-\theta} - (1 + {t_1}^c)^{-\theta}]^{k_2} [1 + {t_1}^c]^{-\theta k_3} \quad (26)$$

Then the posterior of θ is given by

$$\pi\left(\theta \middle| \underline{x}\right) = \frac{H(\underline{x}, \theta)}{\int_0^\infty H(\underline{x}, \theta)\, d\theta} = \frac{\sum_{j=0}^{k_1} \sum_{l=0}^{k_2} (-1)^{j+l} \theta^{\alpha-1} e^{-\theta A_{jl}}}{\Gamma(\alpha) \sum_{j=0}^{k_1} \sum_{l=0}^{k_2} \frac{(-1)^{j+l}}{{A_{jl}}^\alpha}} \qquad (27)$$

where

$$A_{jl} = \beta + (l + k_3) \ln(1 + {t_1}^c) + (j + k_2 - l)\, ln(1 + {t_0}^c)$$

Under squared error loss, the Bayes estimator of the parameter $\theta$of Burr XII distribution is the mean of the posterior of θ is given by

$$\hat{\theta}_5 = E\left(\theta \middle| \underline{y}\right) = \int_0^\infty \theta\, \pi\left(\theta \middle| \underline{y}\right) d\theta = \alpha \frac{\sum_{j=0}^{k_1} \sum_{l=0}^{k_2} \frac{(-1)^{j+l}}{{A_{jl}}^{\alpha+1}}}{\sum_{j=0}^{k_1} \sum_{l=0}^{k_2} \frac{(-1)^{j+l}}{{A_{jl}}^{\alpha}}} \qquad (28)$$

And the Bayes estimator of reliability is

$$\hat{R}_5 = \int_0^\infty [1 + t^c]^{-\theta} \pi\left(\theta \middle| \underline{y}\right)\, d\theta = \frac{\sum_{j=0}^{k_1} \sum_{l=0}^{k_2} \frac{(-1)^{j+l}}{[A_{jl} + \ln(1 + t^c)]^\alpha}}{\sum_{j=0}^{k_1} \sum_{l=0}^{k_2} \frac{(-1)^{j+l}}{{A_{jl}}^\alpha}} \qquad (29)$$

4 Conclusion

To compare between the five different sampling plans by MSE we take

Case (1): Complete sample.

Case (2): Type I censoring.

Case (3): Partial ordering I.

Case (4): Type I censoring.

Case (1): Partial ordering II.

where c known and θ unknown.

To compare the results obtained in each sampling plan, we take (1)$\alpha = 2,\ \beta = 3$ (2) $\alpha = 2,\ \beta = 6$ and generate a value of θ from Gamma distribution and then generate random sample of size $10, 20, 40$ from Burr XII when $c = 2$ and generated $\theta = 0.6414, 0.3417$. We repeated the simulation 1000 times. For each sampling plan we computed Bayes estimator $\hat{\theta}\ of\ \theta,\ \hat{R}\ of\ R,$ and MSE for each. Stopping $t_0 = 2, t_1 = 5, and\ t = 3$ for reliability.

The following notation were used

$\overline{\hat{\theta}_i}$ ≡means of Bayes estimator of θ in the i –th sampling plan. $i.e.,$

$$\overline{\hat{\theta}_i} = \frac{1}{1000}\sum_{j=1}^{1000}\hat{\theta}_{ij}$$

and

$$MSE = \frac{1}{1000}\sum_{j=1}^{1000}(\widehat{\theta}_{ij} - \theta_{(given)})^2$$

$\overline{\hat{R}_i}$ ≡mean of Bayes estimator of R in the i –th sampling plan. $i.e.,$

$$\overline{\hat{R}_i} = \frac{1}{1000}\sum_{j=1}^{1000}\hat{R}_{ij}$$

and

$$MSE = \frac{1}{1000}\sum_{j=1}^{1000}(\widehat{R}_{ij} - R_{(given)})^2$$

The results are given in tables (1) – (4). We write $case\ (i) < case(j)$ to mean MSE of the estimator in $case\ (i)$ < MSE of the estimator in $case(j)$. Prior to $\theta\ is\ G\left(\alpha,\frac{1}{\beta}\right)$.

Table (1)

$\alpha = 2, \beta = 3, c = 2, \theta = 0.6414, t_0 = 2, and\ t_1 = 5$										
n	$\overline{\hat{\theta}_1}$	MS	$\overline{\hat{\theta}_2}$	MS	$\overline{\hat{\theta}_3}$	MS	$\overline{\hat{\theta}_4}$	MS	$\overline{\hat{\theta}_5}$	MS
10	0.6792	0.0491	0.6373	0.0172	0.6079	0.0554	0.6970	0.0612	0.6237	0.0421
20	0.6521	0.0135	0.6141	0.0452	0.5938	0.0521	0.6503	0.0116	0.6124	0.0382
30	0.6458	0.0091	0.5918	0.0515	0.5911	0.0491	0.6521	0.0085	0.6015	0.0214

$n = 10: case\ 2 < case\ 5 < case\ 1 < case\ 3 < case\ 4$
$n = 20: case\ 1 < case\ 4 < case\ 5 < case\ 2 < case\ 3$
$n = 30: case\ 4 < case\ 1 < case\ 5 < case\ 3 < case\ 2$

Table (2)

$\alpha = 2, \beta = 3, c = 2, \theta = 0.6414, t = 3, R = 0.2283, t_0 = 2, and\ t_1 = 5$										
n	$\overline{\hat{R}_1}$	MS	$\overline{\hat{R}_2}$	MS	$\overline{\hat{R}_3}$	MS	$\overline{\hat{R}_4}$	MS	$\overline{\hat{R}_5}$	MS
10	0.2035	0.0622	0.2223	0.0479	0.2537	0.0903	0.2510	0.0882	0.2168	0.0515
20	0.2138	0.0418	0.2615	0.0603	0.1983	0.0537	0.2632	0.0925	0.2001	0.0525
30	0.2135	0.0401	0.2661	0.0831	0.2235	0.0452	0.2385	0.0511	0.2458	0.0724

$n = 10: case\ 2 < case\ 5 < case\ 1 < case\ 4 < case\ 3$
$n = 20: case\ 1 < case\ 5 < case\ 3 < case\ 2 < case\ 4$
$n = 30: case\ 1 < case\ 3 < case\ 4 < case\ 5 < case\ 2$

Table (3)

$\alpha = 2, \beta = 6, c = 2, \theta = 0.3417, t_0 = 2, and\ t_1 = 5$										
n	$\overline{\hat{\theta}_1}$	MS	$\overline{\hat{\theta}_2}$	MS	$\overline{\hat{\theta}_3}$	MS	$\overline{\hat{\theta}_4}$	MS	$\overline{\hat{\theta}_5}$	MS
10	0.3405	0.0011	0.3465	0.0019	0.3638	0.0125	0.3536	0.0052	0.3622	0.0091
20	0.3563	0.0050	0.3498	0.0037	0.3215	0.0273	0.3512	0.0041	0.3300	0.0180
30	0.32	0.04	0.35	0.00	0.31	0.05	0.34	0.00	0.32	0.04

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	01	39	25	45	88	32	10	10	47	18

$n = 10: case\ 1 < case\ 2 < case\ 4 < case\ 3 < case\ 5$

$n = 20: case\ 2 < case\ 4 < case\ 1 < case\ 5 < case\ 3$

$n = 30: case\ 4 < case\ 2 < case\ 5 < case\ 1 < case\ 3$

Table (4)

$\alpha = 2, \beta = 6, c = 2, \theta = 0.3417, t = 3, R = 0.4553, t_0 = 2, and\ t_1 = 5$										
n	$\overline{\hat{R}_1}$	MS	$\overline{\hat{R}_2}$	MS	$\overline{\hat{R}_3}$	MS	$\overline{\hat{R}_4}$	MS	$\overline{\hat{R}_5}$	MS
10	0.4105	0.0312	0.4889	0.0080	0.4605	0.0013	0.4164	0.0103	0.4823	0.0065
20	0.4219	0.0294	0.4862	0.0072	0.4497	0.0017	0.4134	0.0209	0.4709	0.0051
30	0.4255	0.0235	0.4743	0.0075	0.4364	0.0071	0.4001	0.0471	0.4681	0.0041

$n = 10: case\ 3 < case\ 5 < case\ 2 < case\ 4 < case\ 1$

$n = 20: case\ 3 < case\ 5 < case\ 2 < case\ 1 < case\ 4$

$n = 30: case\ 5 < case\ 3 < case\ 2 < case\ 1 < case\ 4$

References:

(1) Al-Hussaini (1991). A characterization of the Burr type XII distribution. Applied Mathematics Letters 4(1), 59-61.

(2) Hazaimeh, H. M. (1998). Bayes estimation of the parameters for the Burr type XII distribution under several sampling plans (Master thesis, Yarmouk University).

(3) McDonald and Richards (1987). Model selection some generalized distribution. Communication in Statistics Theory and Method 16(4), 1049-1074.

(4) Rodriguez R. N. and Taniguchi (1980). A new statistical model for predicting customer octane satisfaction using trained-rater observation. SAE Technical Paper Series 801356. Society of Automotive Engineers. Inc. Oct.

(5) Shah (1993). Estimation of parameters for the Burr type XII distribution moments and marginal densities. Journal of the American Statistical Association. 81(393), 82-86.

(6) Wingo, D. R. (1983). Maximum likelihood methods for fitting the Burr type XII distribution to life test data. Biometric Journal, 25, 77-84.